In conclusion, the sections of the present paper may be summarised as follows:—

1. The formal investigation of artificial double refraction in colloidal solutions as due to a deformation of the medium consisting of a change in the packing of the colloidal particles.

2. The possibility that such deformation may be produced by mechanical stress as arising from the possession of a certain amount of rigidity by such solutions.

3. The analogy between the effects so produced and the double refraction due to a magnetic field.

On the Simple Group of Order 25920.

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To discuss the properties of a group of finite order some concrete form of representation of the group is necessary, except perhaps in the simplest cases. What are called the abstract defining relations (viz., a system of relations of the form

$$A^i = 1, \quad B^j = 1, \quad \ldots\ldots$$
$$A^p B^q \ldots = 1, \quad A^{p'} B^{q'} \ldots = 1, \quad \ldots\ldots$$

between a system of non-commutative symbols A, B, ..., which are necessary and sufficient to ensure that only a finite number of distinct products can be formed from them) no doubt contain implicitly in the most concise form all the properties of the group. To establish the properties, however, on this basis is not in general practicable. For every group there are an infinite variety of possible concrete representations; and in general for an adequate discussion of the properties of the group several of them have to be made use of. In a limited class of cases, including, however, several groups of great importance in analysis, a representation as a group of space-collineations is available. In all such cases it would be expected that this form of representation, as affording scope for space-intuition, would certainly be one of those chosen for discussion. Except, however, as regards the so-called groups of the regular solids, *i.e.*, groups of rotations round a point, this has not been done.

It is proposed in this memoir to discuss the simple group of order 25920 entirely from the point of view of projective geometry. The existence of a group of collineations of this order is not assumed, but is shown to follow from the existence of a remarkable configuration of points, lines and planes in space. This configuration itself arises naturally in connection with a much less complex group of space-collineations. The method followed throughout is synthetical and constructive. To avoid unduly burdening the earlier part of the memoir, it is assumed that the projective groups of finite order on the straight line have been established (as they can be) without appeal to analysis. Further, the simpler properties of the permutation-groups of 4, 5 and 6 symbols are taken as known.

The simple group of order 25920 has formed the subject of very many investigations; but with a single exception the present memoir is not directly connected with any of them, except of course as regards subject-matter and the better known results. The single exception is the inaugural dissertation of Herr A. Witting.* In his work Herr Witting starts from the simple group of order 25920, defined as a group of collineations by the analytical form of its five generating substitutions, and deduces from these premises the existence of the above-mentioned configuration of points, lines and planes. As stated above, the course followed here is to establish independently the existence of the configuration by considerations of projective geometry, and to deduce the existence and properties of the group.

The Abelian group, of order 16, of collineations of order 2, and the group of order 16·720 that contains it self-conjugately, with a discussion of which this investigation commences, are familiar in their analytical form to geometers in connection with a family of Kummer surfaces that possess 16 assigned nodes. In particular the existence of the larger group of order 16·720 has been made immediately obvious in a memoir of Professor Klein† by using a suitably chosen system of line-co-ordinates. For the purposes of this memoir it has, however, been necessary to establish the existence of these groups, and the nature of the configuration of 15 line-pairs which is invariant for them, directly from purely geometrical considerations, or the later developments would have been unintelligible.

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* A Witting, 'Über eine Configuration in Raume auf welche die Transformationstheorie der Hyperelliptischen Functionen ($p = 2$) führt.' Gottingen, 1887. I had no knowledge of Herr Witting's work till after the whole of the present memoir had been written.

† F. Klein, 'Mathematische Annalen,' vol. 2, p. 198.

I. *On the Abelian* G_{16}.

1. A projective transformation of order 2 of a straight line has two distinct fixed points (real or imaginary). If these are A and A′, any second projective transformation of order 2 of the line which is permutable with the previous one must permute A and A′. The most general group of projective transformations of the line whose operations are all of order 2, and all permutable with each other, must be a G_4. This contains, besides identity, three operations of order 2, such that each permutes the two fixed points of the other two. If A, A′ ; B, B′ ; C, C′, are the three pairs of fixed points, each range ABA′B′, ACA′C′, BCB′C′ is harmonic; and apart from projective transformations, the configuration of the three pairs of points is absolutely determinate. Not more than two of the pairs can be real; but the distinction between real and imaginary points is, for the present purpose, unessential.

A projective transformation of order 2 of space must leave either every point of a plane ABC and some one point O not in the plane fixed; or it must leave every point of two non-intersecting lines AB and CD fixed. In each case the transformation is completely determined by its fixed points. In the first case, if P is any point, and if OP meets the plane ABC in *o*, then P is changed into *p*, lying on OP*o*, so that OP*op* is harmonic. In the second case, if a line through P meets AB and CD in Q and *q*, then P is changed into *p*, lying on QP*q*, so that QP*qp* is harmonic.

In the first case any projective transformation which leaves O and the plane ABC unchanged is permutable with the given transformation of order 2. In the second case any projective transformation which either leaves unchanged or permutes AB and CD is permutable with the given transformation.

If two transformations of the first kind (or perspectives) are permutable, the fixed point of each must be in the fixed plane of the other; so that A and B may be taken as the fixed points, and BCD, CDA as the fixed planes. If a third perspective is permutable with each of these, its fixed point must be in CD, and its fixed plane must pass through AB. Hence C may be taken as its fixed point, and ABD as its fixed plane. Then the perspective of which D is the fixed point, and ABC the fixed plane (which is the product of the three former ones), is permutable with each of them; and it is obvious that no other perspective can be permutable with each of the four. The products of these perspectives two and two are the three projective transformations of order 2 for which AB, CD; AC, BD; AD, BC are the pairs of fixed lines, and these, with the four perspectives and identity, constitute an Abelian group of order 8, whose operations are of order 2.

Now in any Abelian group of collineations of order 2 the number of perspectives, if any, must be half the order of the group; and therefore, if the order of the group is greater than 8, it can contain no perspectives.

2. I proceed then to consider the Abelian groups of collineations of order 2 which contain no perspectives. If L_1 and L_2 are the fixed lines of one of the collineations of such a group, every collineation of the group must either permute L_1 and L_2, or must leave each of them unchanged. Moreover, the fixed lines of a collineation which leaves both L_1 and L_2 unchanged must meet both L_1 and L_2; and the fixed lines of a collineation which permutes L_1 and L_2 meet neither of them.

Further, the only collineation, other than identity, of the group which leaves every point on L_1 unchanged, is that of which L_1 and L_2 are the fixed lines; for a collineation of order 2 which leaves every point on one line and just two points on some non-intersecting line unchanged is necessarily a perspective.

Now it has been seen that the greatest Abelian group of projective transformations of order 2 of a straight line is a G_4; hence the greatest Abelian group of collineations of order 2 which changes each of two given non-intersecting lines into itself has order not exceeding 8; and the greatest such group which either permutes or changes into themselves two given non-intersecting lines has order not exceeding 16.

That such a G_{16} actually exists may be shown by the following construction. Let L_1, L_2 be any two non-intersecting lines, and on L_1 take three pairs of points A, A'; B, B'; C, C', so that each range $ABA'B'$, $BCB'C'$, $CAC'A'$ is harmonic. Let M_1 and M_2 be the fixed lines of a collineation of order 2 which permutes L_1 and L_2, and denote by a, a'; b, b'; c, c' the points on L_2 into which this collineation changes A, A'; B, B'; C, C'.

Any collineation of order 2 (not a perspective) is completely specified by its two fixed lines; so that $[L_1, L_2]$ may be used to denote the collineation of order 2 which leaves every point of L_1 and of L_2 unchanged.

With this notation $[L_1, L_2]$ is clearly permutable with

$$[Aa, A'a']; \quad [Aa', A'a]; \quad [Bb, B'b']; \quad [Bb', B'b]; \quad [Cc, C'c']; \quad [Cc', Cc],$$

since the fixed lines of the six last written collineations meet both L_1 and L_2. Moreover, these six collineations are permutable among themselves; in fact, $[Aa, A'a']$ leaves the lines Aa' and $A'a$ unchanged and permutes the pairs Bb and $B'b'$, Bb' and $B'b$, Cc and $C'c'$, Cc' and $C'c$. Hence the above seven collineations with identity form a set of eight permutable collineations each of which leaves both L_1 and L_2 unchanged. But it is shown above that the order of an Abelian group of collineations of order 2, which leaves L_1 and L_2 unchanged, cannot exceed 8. Hence,

$$1;\ [L_1, L_2];\ [Aa, A'a'];\ [Aa', A'a];\ [Bb, B'b'];\ [Bb', B'b];\ [Cc, C'c'];\ [C'c, Cc']$$

constitute an Abelian G_8 whose operations are all of order 2, for which each of the lines L_1 and L_2 is unchanged. This G_8 does not contain $[M_1, M_2]$ which permutes L_1 and L_2; Aa' and $A'a$; Bb' and $B'b$; Cc' and $C'c$, and leaves unchanged Aa, $A'a'$, Bb, $B'b'$, Cc, $C'c'$. Hence $[M_1, M_2]$ is not contained in the G_8, and is permutable with each of its operations. Therefore G_8 and $[M_1, M_2]$ generate an Abelian G_{16}, whose operations are all of order 2 and either permute or leave unchanged L_1 and L_2. The existence of the group is thus proved.

Returning now to the construction, the essential elements of it are the lines L_1, L_2, M_1 and the points A, A′, B; in fact, from A, A′, B the other three points B′, C, C′ on L_1 are constructed by harmonic section. If from these six points lines are drawn to meet L_2 and M_1, the points in which they meet L_2 are a, a', b, b', c, c'. Further, if Aa, $A'a'$ meet M_1 in P and P′, and if $APap$, $A'P'a'p'$ are harmonic, then pp' is M_2.

Now by a suitable projective transformation any three non-intersecting lines may be changed into L_1, L_2, M; and by a subsequent transformation which leaves these three lines unchanged, any three points on L_1 may be changed into A, A′ and B. Hence any two Abelian G_{16}'s whose operations are all collineations of order 2 are conjugate within the general group of collineations.

II. *The Invariant Configuration of the* G_{16}.

3. I consider now the configuration formed by the 15 pairs of fixed lines of the 15 collineations, other than identity, contained in such a G_{16}. Each line of any pair is unaltered by a G_8, and the two lines are permuted by the other eight collineations making up the G_{16}. Hence each line of a pair is met by both the lines of six other pairs, and is not met by either of the lines of the remaining eight pairs. Consider two pairs, say L_1 and L_2, M_1 and M_2, which do not intersect. Such a pair of pairs can be chosen in 60 ways. Both L_1 and L_2 are unchanged for the collineations of a G_8; and both M_1 and M_2 are unchanged by the operations of another G'_8. These two sub-groups of the G_{16} have a G_4 in common. If R_1, R_2; S_1, S_2; T_1, T_2 are the fixed lines of the collineations, other than identity, in this G_4, each of them must meet L_1, L_2, M_1 and M_2. Since $[L_1, L_2]$ and $[M_1, M_2]$ both leave R_1, R_2, S_1, S_2, T_1, T_2 unchanged, so also does the collineation which is the product of $[L_1, L_2]$ and

$[M_1, M_2]$. Hence if N_1 and N_2 are the fixed lines of this collineation, the six pairs

$$L_1, L_2;\ M_1, M_2;\ N_1, N_2;$$

and

$$R_1, R_2;\ S_1,\ S_2;\ T_1, T_2;$$

lie on a quadric, the first three belonging to one system of generators and the last three to the other. If R_1, R_2 meet L_1 in A and A' and L_2 in a and a', then, as seen above, among the collineations of the G_{16} which leaves both L_1 and L_2 unchanged, is that for which Aa' and $A'a$ are the fixed lines. Hence the fixed lines of the remaining nine collineations (which do not lie on the quadric considered) are the pairs of lines which join across the intersections of any one of the pairs

$$L_1, L_2;\ M_1, M_2;\ N_1, N_2;$$

with any one of the pairs

$$R_1, R_2;\ S_1,\ S_2;\ T_1, T_2.$$

The quadric in question is completely determined by any non-intersecting pair of pairs that lie on it. Of these there are six, while it has been seen that out of the 15 pairs a non-intersecting pair of pairs can be chosen in 60 ways. Hence there are just 10 quadrics on which the 15 pairs of lines lie in sets of six. Moreover, the 15 pairs of lines form the complete intersections of the quadrics in the sense that any two of the quadrics intersect in two pairs of the lines. Thus, of the eight pairs of lines which do not intersect L_1 and L_2, the quadric considered contains two. There are therefore three other quadrics which contain L_1 and L_2. So there are three others which contain M_1 and M_2, and three others which contain N_1 and N_2. Similarly, each of the other nine quadrics must contain either R_1 and R_2, or S_1 and S_2, or T_1 and T_2. The quadric considered must therefore intersect any one of the remaining nine in one pair from L_1 and L_2, M_1 and M_2, N_1 and N_2, and one pair from R_1 and R_2, S_1 and S_2, T_1 and T_2.

4. If instead of starting with a non-intersecting pair of pairs such as L_1 and L_2, M_1 and M_2, one considers the intersecting pair L_1 and L_2, R_1 and R_2, the G_4 which leaves unchanged each of the four lines consists, with identity, of $[L_1, L_2]$, $[R_1, R_2]$ and their product. The fixed lines of their product are the lines joining across the intersections of L_1, L_2, R_1 and R_2; so that the three pairs of fixed lines of the three collineations which, with identity, constitute this G_4, are the pairs of opposite sides of a tetrahedron. Since there are six collineations which leave both L_1 and L_2 unchanged, and two of these occur with L_1 and L_2 in the G_4 in question, there are two other tetrahedra each of which have L_1 and L_2 for a pair of opposite sides; and therefore in all 15 such tetrahedra can be formed from the 15 pairs of lines.

5. Another remarkable property of the 15 pairs of lines is that just six sets of five pairs each may be selected from them so that no two pairs of any set of five intersect; thus L_1 and L_2, M_1 and M_2, and the three pairs which join across the intersections of N_1 and N_2 with R_1 and R_2, S_1 and S_2, T_1 and T_2 respectively form such a set of five. It is, in fact, obvious that no one of the last three pairs intersects L_1 and L_2 or M_1 and M_2. Call P_1 and P_2 the lines which join across the intersections of N_1 and N_2 with R_1 and R_2, and Q_1 and Q_2 those which join the intersections of N_1 and N_2 with S_1 and S_2. If P_1 and P_2 intersect Q_1 and Q_2, then $[P_1, P_2]$ would leave unchanged N_1 and Q_1, so that their point of intersection, *i.e.*, the point of intersection of N_1 and S_1, would be a fixed point for $[P_1, P_2]$. But the only fixed points on N_1 for $[P_1, P_2]$ are the points where R_1 and R_2 meet N_1, so that the point of intersection of N_1 and S_1 is not a fixed point for this collineation. No two of the set of five lines in question then intersect, and they are the only set containing L_1, L_2, M_1, M_2 of which this is true.

Now there are 60 non-intersecting pairs of pairs, from each of which such a set of five will arise, and since any one such set may arise from any one of the $\frac{1}{2}\cdot 5\cdot 4$ pairs contained in it, the number of sets must be six.

III. *On the* $G_{16\cdot 720}$.

6. I go on now to determine the collineations which leave this set of 15 pairs of lines invariant. It may be first noted that the only collineations which change each pair into itself are those of the G_{16}. In fact, the square of any such collineation transforms each line of each pair into itself, and since each pair is met by six others, it must transform every point of each line of each pair into itself. It is, therefore, the identical transformation, and the collineation itself is, therefore, one of order 2. But it has been shown that all collineations of order 2, which are permutable with each collineation of the G_{16}, must be contained in the G_{16}.

It remains only to consider those collineations which permute some or all of the pairs among themselves, and I shall first determine those that permute six pairs lying on one of the 10 quadrics among themselves, say the pairs

$$L_1, L_2;\quad M_1, M_2;\quad N_1, N_2;$$

and

$$R_1, R_2;\quad S_1, S_2;\quad T_1, T_2.$$

Any projective transformation which permutes these among themselves necessarily permutes the remaining nine pairs (which join across the points of intersection of the six) among themselves, and also transforms the quadric containing the six pairs into itself. Now the projective transformations of space, which leave a quadric invariant, consist of—

(i) Transformations which leave every one of the first set of generators unchanged, while they transform the second set exactly as the points on a line are transformed by the projective transformations of the line; combined independently with those which leave every one of the second set of generators unchanged, and similarly transform the first set; and

(ii) The previous set of transformations compounded with a transformation of order 2 which permutes the two sets of generators.

The most general projective group on a straight line, for which three pairs of points AA′, BB′, and CC′, such that ABA′B′, BCB′C′, CAC′A′ are harmonic, are permuted among themselves, is the well-known G_{24}, ordinarily called the octahedral group.

Hence the most general group for which the three pairs L_1 and L_2, M_1 and M_2, N_1 and N_2 are permuted among themselves, while each of the six lines R_1, R_2, S_1, S_2, T_1, T_2 is unchanged is a G_{24}. Similarly there is a G'_{24} which leaves each of the lines L_1, L_2, M_1, M_2, N_1, N_2 unchanged and permutes the pairs R_1 and R_2, S_1 and S_2, T_1 and T_2 among themselves. It is obvious that the G_{24} and G'_{24} have no common collineations except identity, and that every collineation of G_{24} is permutable with every one of G'_{24}.

Suppose that U_1 and U_2 are the lines which join across the intersections of L_1, L_2 with R_1, R_2, and V_1 and V_2 those that join across the intersections of M_1, M_2 with S_1, S_2, U_1 passing through the intersection of L_1 and R_1, and V_1 through the intersection of M_1 and S_1. Then V_1 and V_2 do not intersect either of the pairs L_1 and L_2, or R_1 and R_2. Hence $[V_1, V_2]$ permutes L_1 and L_2, and also R_1 and R_2; and therefore $[V_1, V_2]$ leaves U_1 and U_2 unchanged. Hence U_1, U_2 and V_1, V_2 are intersecting pairs. Let V_2 and U_2 intersect in O, and consider the perspective of order 2 of which O is the fixed point and the plane of U_1 and V_1 the fixed plane. If V_1 meets U_2 in o, then O, o divide harmonically the two points in which U_2 is met by L_1 and L_2 (or, which is the same thing, by R_2 and R_1). Hence the perspective permutes L_1 with R_1, and L_2 with R_2. Similarly it permutes M_1 with S_1 and M_2 with S_2. The perspective in question, therefore, replaces the two pairs L_1 and L_2, and M_1 and M_2 by the two pairs R_1 and R_2, and S_1 and S_2. Hence it must replace the pair N_1 and N_2, which forms with each of the pairs L_1 and L_2 or M_1 and M_2 a harmonic range on any transversal, by the pair T_1 and T_2, which is similarly related to R_1 and R_2, and S_1 and S_2.

The perspective therefore permutes the two sets of generators on the quadric, and with the G_{24} and G'_{24} generates a G_{1152}, which permutes the six pairs under consideration among themselves. This group of collineations, which leaves one of the 10 quadrics unchanged, permutes the remaining 9 transitively among themselves. In fact it obviously contains collineations which change any one of the intersecting pairs of pairs, such as L_1, L_2, R_1, R_2, into any other; and the nine intersecting pairs of pairs are, as shown above, the intersections of the invariant quadric with the other nine. There is a similar G_{1152} which leaves any one of the other quadrics invariant. Hence, since the 10 quadrics are transitively permuted by the whole group of collineations for which the 15 pairs of lines are invariant, the order of this group is not less than 11520.

Every collineation which leaves the 15 pairs invariant must permute among themselves the six sets of five non-intersecting pairs that can be formed from them. Each pair occurs in two and only two of these sets. Hence every collineation which leaves each of the six sets unchanged must leave each of the 15 pairs unchanged, and it is shown above that the only collineations for which this is the case are those of the G_{16}. The G_{16} is therefore a self-conjugate sub-group of the total group of collineations for which the 15 pairs of lines is invariant; and in respect of this sub-group the group is isomorphic with a group of permutations of six symbols. Now

$$11520 = 16 \times 720,$$

and 720 is the order of the greatest group which is simply isomorphic with a permutation group of six symbols.

Finally, therefore, the order of the greatest group of collineations for which the 15 pairs of lines forms an invariant configuration is 11520. The group contains a self-conjugate Abelian sub-group of order 16, whose operations are all of order 2; and the factor-group in respect of this self-conjugate sub-group is simply isomorphic with the symmetric group of degree 6.

7. If, in addition to collineations, dualistic transformations of space are admitted, the total group of space-transformations for which the configuration of 15 pairs of lines is invariant, is a group of order 23040, constituted by combining with the group of 11520 collineations a set of 11520 dualistic transformations.

A reciprocation with respect to the quadric containing L_1, L_2, M_1, M_2, N_1, N_2, R_1, R_2, S_1, S_2, and T_1, T_2 is, in fact, a dualistic transformation of order 2, which leaves unchanged each of the lines of the six pairs on the quadric and permutes the lines of each of the remaining nine pairs. This transformation, combined with the G_{16}, gives a G_{32}, Abelian, and having all its operations of order 2, for which each of the 15 pairs is invariant. The 16 dualistic transformations of this G_{32} are obtained by combining each of the collineations of the G_{16} with the above reciprocation. The nine transformations that arise by combining the collineations whose fixed lines do not lie on the first quadric with the reciprocation are immediately seen to be reciprocations with respect to the other nine quadrics. The remaining six transformations arise by combining $[L_1, L_2]$; $[M_1, M_2,]$ ... ; $[T_1, T_2]$ with the reciprocation. Now, the effect of $[L_1, L_2]$, followed by the reciprocation, is to permute the lines forming each of five pairs, viz., M_1 and M_2, N_1 and N_2, and the lines joining across the intersections of L_1 and L_2 with R_1 and R_2, S_1 and S_2, T_1 and T_2 respectively; and to leave each of the lines forming the other 10 pairs unchanged. The set of five pairs just written is one of the six sets of five, considered in § 5. Hence, each of the remaining six dualistic transformations permutes the lines of each pair of one of the six sets of five, and leaves unchanged the lines of the remaining 10 pairs. This G_{32} is contained self-conjugately within the group of space transformations which permutes the 15 pairs among themselves; so that, as stated above, the order of this total group must be 23040.

IV. *On the* $G_{16 \cdot 60}$.

8. The symmetric group of six symbols has two distinct sets of conjugate sub-groups, isomorphic with the alternating group on five symbols, of order 60. A sub-group of one of these sets permutes the six symbols transitively, one of the other set leaves one symbol unchanged and permutes the other five transitively. Hence the $G_{16 \cdot 720}$, which contains the Abelian G_{16} self-conjugately, has two distinct sets of sub-groups of order 16·60, each of which contains the G_{16}, and in respect of it is isomorphic with the alternating group on five symbols. A sub-group of one set permutes transitively among themselves the six sets of five non-intersecting pairs, which can be formed from the fixed lines of the 15 collineations of order 2 contained in the G_{16}. A sub-group of the other set leaves one of the set of five non-intersecting pairs unchanged, and permutes the remaining five.

It is a $G_{16 \cdot 60}$ of this second set that is now to be considered in some detail, as leading up to the space-configuration of points, lines, and planes. Let L_1, L_2; M_1, M_2; P_1, P_2; Q_1, Q_2; R_1, R_2 be the five non-intersecting pairs which are permuted by every collineation of the $G_{16 \cdot 60}$ selected. Every collineation of the $G_{16 \cdot 60}$ is completely specified by the permutation which it gives of these 10 lines. For if the collineations S and T gave the same permutation, then ST^{-1} would leave each of the 10 lines, and therefore, also, each of the fixed lines of the remaining 10 collineations of order 2 of the G_{16} unchanged; and it has been seen above that such a collineation leaves every point of space unchanged.

The permutations corresponding to

$$[L_1, L_2], \;\ldots\ldots\ldots,\; [R_1, R_2],$$

are

$$\begin{array}{lllll}
 & (M_1\,M_2) & (P_1\,P_2) & (Q_1\,Q_2) & (R_1\,R_2), \\
(L_1\,L_2) & & (P_1\,P_2) & (Q_1\,Q_2) & (R_1\,R_2), \\
(L_1\,L_2) & (M_1\,M_2) & & (Q_1\,Q_2) & (R_1\,R_2), \\
(L_1\,L_2) & (M_1\,M_2) & (P_1\,P_2) & & (R_1\,R_2), \\
(L_1\,L_2) & (M_1\,M_2) & (P_1\,P_2) & (Q_1\,Q_2) & \;;
\end{array}$$

and hence those corresponding to the other 10 collineations of order 2 of the G_{16} are

$$(L_1\,L_2)\,(M_1\,M_2),\quad (L_1\,L_2)\,(P_1\,P_2),\quad (L_1\,L_2)\,(Q_1\,Q_2),\quad (L_1\,L_2)\,(R_1\,R_2),$$
$$(M_1\,M_2)\,(P_1\,P_2),\quad (M_1\,M_2)\,(Q_1\,Q_2),\quad (M_1\,M_2)\,(R_1\,R_2),$$
$$(P_1\,P_2)\,(Q_1\,Q_2),\quad (P_1\,P_2)\,(R_1\,R_2),$$
$$(Q_1\,Q_2)\,(R_1\,R_2).$$

A sub-group of order 60, isomorphic with the alternating group on five symbols, must give all the even permutations of the five pairs of non-intersectors. Hence an operation of order 3 of such a sub-group leaves each line of two pairs unchanged; *i.e.*, as a permutation of the 10 lines, it leaves four fixed and permutes the other six in two cycles of three each. Now, if the sub-group of order 60 permuted the 10 lines transitively, an operation of order 3 belonging to it would only leave one unchanged; and therefore it must permute them intransitively in two sets of five each. Let

$$L_1,\, M_1,\, P_1,\, Q_1,\, R_1;\quad L_2,\, M_2,\, P_2,\, Q_2,\, R_2;$$

be the two sets for one sub-group G_{60} of order 60. Then the permutations of the 10 lines given by the collineations of the G_{60}, are formed by taking any even permutation of the lines of one set and combining it with the corresponding even permutation of those of the other set. In particular the G_{60} is generated by

$$(L_1\,M_1\,P_1\,Q_1\,R_1)\,(L_2\,M_2\,P_2\,Q_2\,R_2),$$
$$(L_1\,M_1)\,(P_1\,Q_1)\,(L_2\,M_2)\,(P_2\,Q_2);$$

and its tetrahedral sub-group which leaves L_1 and L_2 unchanged by

$$(M_1\,P_1)\,(Q_1\,R_1)\,(M_2\,P_2)\,(Q_2\,R_2)$$

and

$$(P_1\,Q_1\,R_1)\,(P_2\,Q_2\,R_2).$$

Further, since the Abelian G_{16} and the G_{60} have no collineation, except the identical one in common, the 960 collineations of the group are given by combining any collineation of the G_{16} with any collineation of the G_{60}.

It may further be noticed that since the $G_{16\cdot 60}$ has no self-conjugate sub-group of index 2, no collineation of order 2 belonging to it can be a perspective.

V. *On the Space-Configuration.*

9. The line L_1 is invariant for the sub-group G_8 of the Abelian G_{16}, which consists of

$$1,\ (M_1\ M_2)\ (P_1\ P_2),\ (M_1\ M_2)\ (Q_1\ Q_2),\ (M_1\ M_2)\ (R_1\ R_2),\ (P_1\ P_2)\ (Q_1\ Q_2),$$
$$(P_1\ P_2)\ (R_1\ R_2),\ (Q_1\ Q_2)\ (R_1\ R_2),\ (M_1\ M_2)\ (P_1\ P_2)\ (Q_1\ Q_2)\ (R_1\ R_2),$$

and for the above tetrahedral G_{12}, generated by

$$(M_1\ P_1)\ (Q_1\ R_1)\ (M_2\ P_2)\ (Q_2\ R_2),$$
$$(P_1\ Q_1\ R_1)\ (P_2\ Q_2\ R_2).$$

These generate a $G_{8\cdot 12}$ for which L_1 is invariant; and this $G_{8\cdot 12}$ contains a G'_{12}, generated by

$$(P_1\ P_2)\ (Q_2\ Q_2),$$
$$(P_1\ Q_1\ R_1)\ (P_2\ Q_2\ R_2),$$

which on L_1 obviously sets up a tetrahedral projective group. Hence the $G_{8\cdot 12}$, for which L_1 is invariant, is multiply isomorphic with a tetrahedral group, and sets up on L_1 a group of projective transformations which contains a tetrahedral group. But the only finite group of projective transformations of a straight line which contains a tetrahedral group, and is simply or multiply isomorphic with a tetrahedral group, is the tetrahedral group itself. Hence the group of projective transformations on L_1, which the $G_{8\cdot 12}$ that leaves L_1 invariant sets up, is a tetrahedral group. This tetrahedral group of projections on L_1 has four sub-groups of order 3; and for each there is a pair of fixed points. If these pairs are a, a'; b, b'; c, c'; d, d'; then, suitably choosing one from each pair, a, b, c, d, are a set of points which are transitively permuted among themselves by the $G_{8\cdot 12}$ for which L_1 is invariant; as also are a', b', c', d'. Both a and a' are therefore invariant for a sub-group of order 24; and each is therefore one of a set of 40 points which are transitively permuted by the $G_{16\cdot 60}$. The plane through L_2 and a is one of a set of 40 planes which are transitively permuted by the $G_{16\cdot 60}$, and so also is the plane through L_2 and a'.

The set of 40 points arising by the collineations of the group from a, and the set of 40 planes arising from L_2a', are the points and planes of the space configuration.

10. No four lines, such as M_1, P_1, Q, R_1 belonging to four distinct pairs can lie on a quadric. For $(L_1\ L_2)\ (M_1\ M_2)$ being a permutation of the 10

lines given by one of the collineations of the group, it follows that M_2 and therefore also P_2, Q_2, R_2, would lie on the same quadric, which is not the case. There are therefore just two lines meeting M_1, P_1, Q_1, R_1; say S_1 and S_1'; and the configuration of the five non-intersecting pairs, as given in § 5, shows that neither S_1 nor S_1' meets L_1, L_2, M_2, P_2, Q_2, or R_2. The collineation represented by $(P_1\, Q_1\, R_1)\,(P_2\, Q_2\, R_2)$ changes this pair of lines, and therefore each of them, into itself. The point in which S_1 meets M_1 is therefore a fixed point for the collineation $(P_1\, Q_1\, R_1)\,(P_2\, Q_2\, R_2)$. Similarly the points in which S_1 meets P_1, Q_1, R_1 respectively are fixed points for the collineations

$$(Q_1\, R_1\, M_1)\,(Q_2\, R_2\, M_2),\ (R_1\, M_1\, P_1)\,(R_2\, M_2\, P_2),\ (M_1\, P_1\, Q_1)\,(M_2\, P_2\, Q_2),$$

and the four points are permuted transitively among themselves by the tetrahedral group which these collineations of order 3 generate. They must therefore belong either to the set of 40 points which arise from a or to the set which arises from a'. But the two points in which S_1 and S_1' meet M_1 are the fixed points of $(P_1\, Q_1\, R_1)\,(P_2\, Q_2\, R_2)$ which lie on M_1, one of which belongs to the set a, and the other to the set a'. Hence, S_1 may be taken to contain four points of the set a, and S_1' to contain four points of the set a'. Now, the greatest sub-group of the $G_{16\cdot 60}$ for which S_1 is invariant is the above tetrahedral group permuting M_1, P_1, Q_1, R_1. Hence S_1 is one of 80 lines permuted by the $G_{16\cdot 60}$, and on each of them there lie just four points of the set a. These 80 lines with L_1, L_2, M_1, M_2, P_1, P_2, Q_1, Q_2, R_1, R_2, form the set of 90 lines belonging to the configuration, each of which passes through four of the 40 points. Moreover, the 80 lines fall into a set of 40 pairs, which are permuted by the $G_{16\cdot 60}$. Such a pair is formed by S_1 and the line S_2 (intersecting M_2, P_2, Q_2, R_2) into which S_1 is changed by the collineation $(M_1\, M_2)\,(P_1\, P_2)\,(Q_1\, Q_2)\,(R_1\, R_2)$. For this pair is invariant for the sub-group formed by combining the tetrahedral group for which S_1 is invariant with the preceding collineation of order 2. The 90 lines thus fall into 45 pairs which are permuted by the group.

Through each of the 40 points there passes just one of the lines L_1, L_2, ..., R_1, R_2. Moreover, the remaining 80 lines, being permuted transitively by the group, and each containing four of the 40 points which are also permuted transitively, 8 of the 80 lines must pass through each point of the 40 points. Hence, in all, just 9 of the 90 lines pass through each of the 40 points.

It may be further noticed that S_1 (and therefore any one of the 80 lines arising from it) contains none of the points of the a' set. For these all lie on the 10 original lines, and the only points in which S_1 meets them belong to the a set.

Further, just 32 of the 80 lines arising from S_1 meet M_1, eight passing through each of the four a-points on M_1. No one of the eight S_1-lines through a given a-point on M_1 meets M_2; and each meets three of the remaining eight lines, L_1, L_2, P_1, P_2, Q_1, Q_2, R_1, R_2 chosen from different pairs. These eight lines contain 32 points of the a-set, and of these, 24 lie on the eight lines of the S_1-set which pass through a given a-point on M_1.

Hence there are just eight out of the 32 a-points lying on L_1, L_2, P_1, P_2, Q_1, Q_2, R_1, R_2, such that the lines joining them to an assigned a-point on M_1, are not S_1-lines. Since the eight lines L_1, L_2 ...... are permuted transitively by the group of collineations which leaves M_1 invariant, these eight points must lie one on each of the eight lines.

Hence, if a is an a-point, and α, β, γ, δ, four a-points lying on one of the 10 lines, which does not form a pair with that containing a; then of the four lines $a\alpha$, $a\beta$, $a\gamma$, $a\delta$, three are S_1-lines, each containing two other a-points, and the fourth is not an S_1-line.

11. Suppose now that a, b, c, d, are the a-points lying on L_1, and that α is an assigned a-point lying on M_1. The plane $L_1\alpha$ contains the S_1-lines joining α to three out of a, b, c, d; say $b\alpha$, $c\alpha$, $d\alpha$. Each of these contains two other a-points; say, β, γ on $b\alpha$; δ, ϵ on $c\alpha$; ζ, η are $d\alpha$. These six points are the intersections of $L_1\alpha$ with P_1, P_2, Q_1, Q_2, R_1, R_2. Since no S_1-line intersects both lines of an original pair, it may be taken that

$$\begin{array}{llllll}\beta, & \gamma, & \delta, & \epsilon, & \zeta, & \eta\end{array}$$

lie respectively on P_1, Q_1, P_2, R_1, Q_2, R_2.

Of the lines joining β to a, c, d, two are S_1-lines, say βa, βc. Each contains two other a-points, not lying on L_1, L_2, M_1, P_1, or P_2, and not being γ. Hence, if βa passes through ϵ, ζ, then βc must pass through η and an a-point θ not belonging to the previous six. This point must, therefore, lie on M_2; and hence the eight points in which $L_1\alpha$ meets M_1, M_2, P_1, P_2, Q_1, Q_2, R_1, R_2, are all a-points. It follows immediately that this plane contains eight S_1-lines: for through each of the eight a-points in the plane, which does not lie on L_1, three such lines can be drawn, and the eight points lie three by three on these lines.

Also through L_1 a set of four planes can be drawn which contain the 32 a-points that do not lie on L_1 or L_2, viz., the four planes through L_1 and the four a-points on M_1 (or on any one of the remaining seven, excluding L_2). These four planes are permuted transitively by the sub-group of order 96 for which L_1 is invariant and therefore, from any one of them, $L_1\alpha$, a set of 40 planes arise which are permuted transitively by the $G_{16\cdot 60}$.

It remains to consider the point a_2 in which $L_1\alpha$ meets L_2. Since it is one of four points permuted by the sub-group for which L_2 is invariant, it must be an a-point, or an a'-point. If it were an a-point, the collineation $[M_1\ M_2]$ which permutes L_1 and L_2 must permute a_2 with one of the a-points on L_1, say a, changing the plane $L_1\alpha$ (which contains a_2) into the plane L_2a and leaving the line aa_2 unaltered. But $[M_1, M_2]$ leaves every point on M_1 and M_2 unaltered and, therefore, leaves the points α, θ in which $L_1\alpha$ meets M_1 and M_2 unaltered. Hence the line $\alpha\theta$ must coincide with aa_2. Similarly the lines ba_2, ca_2, da_2 must each contain one of the pairs β, δ; γ, ζ; ϵ, η. This is certainly impossible, for it implies that the 12 a-points of $L_1\alpha$ (other than a_2), which have been proved to lie four by four on 9 straight lines, also lie three by three on four concurrent lines. Hence a_2 must be an a'-point; and each of the 40 planes arising from $L_1\alpha$ contains just 12 of the 40 a-points. These 12 points lie four by four on nine lines, viz., the eight S_1-lines in the plane, and the one line in the plane which belongs to the original 10, in other words the nine out of the 90 lines which lie in the plane contain the 12 out of the 40 points four by four. The four of the 40 planes which pass through S_1 each contain just eight a-points other than those on S_1. Hence, in all, the four planes contain 36 of the 40 points. The four remaining ones must be permuted among themselves by the collineations of the tetrahedral sub-group of $G_{16\cdot 60}$ for which S_1 is invariant. But the only sets of four a-points which are permuted by this group are those on L_1, L_2, S_1 and its pair S_2. Hence the four planes of the set through S_1 contain all the a-points except those on S_2. They are, therefore, determined by drawing the four planes through S_1 and the four a-points on any other S_1-line which intersects neither S_1 nor S_2. Now it has been seen that the S_1-lines in any of these planes intersect only in a-points. Further, an S_1-line which does not lie in one of these planes does not intersect S_1 at all. Hence the 40 a-points form the complete intersection of the 80 S_1-lines, and therefore also of the whole set of 90 lines of the configuration.

With each of the 40 a-points one of the 40 planes may be associated in a definite manner. Let a, a' be the fixed points of a collineation of order 3 of the $G_{16\cdot 60}$ on L_1, and consider a and the plane L_2a'. If S_1 is an S_1-line through a it must meet L_2a' in a point s which does not lie on L_2. Now the four planes through S_1 which contain all the 40 points except the four that lie on S_2, the pair of S_1, meet L_2a' in four concurrent lines passing through s. But of the 12 a-points in L_2a' not more than nine can lie on four concurrent lines. Hence of the 12 a-points on L_2a', at least three do not lie on the four planes through S_1 which contain all the a-points, except those of S_2. In other words S_2 lies on L_2a'. Similarly the pair of each S_1-line, which passes through a, lies on L_2a'. Also L_1 passes through a, and L_2 lies on L_2a'. Hence the pair to each one of the nine lines through a, belonging to the set of 90, lies on L_2a'. The point s in which S_1 meets L_2a' is not an a-point, because the 4 of the 40 planes which contain S_2 meet S_1 in points which are not a-points. Hence the a-points on L_2a' are distinct from the a-points on the nine lines which pass through a. Now, there are 12 a-points on L_2a', and $1+3\cdot 9=28$ a-points on the nine lines through a. The nine S_1-lines in L_2a' and the nine S_1-lines through a (their pairs) thus contain the whole system of 40 points.

Since the planes and points are permuted transitively by the group $G_{16\cdot 60}$, 12 of the 40 planes pass through each one of the 40 points and contain the nine lines passing through it in sets of three, four of the planes passing through each of the lines.

12. The complete specification of the configuration that has thus been established is as follows:—

It contains 40 points, 90 lines (forming 45 pairs) and 40 planes. Through each point there pass nine of the lines and 12 of the planes. Through every two of these 9 lines, one of the 12 planes passes, containing a third line; so that the planes contain the lines three by three, and the lines are the intersection of the planes four by four.

In each plane lie 12 of the points and nine of the lines, every two lines intersecting in one of the points through which a third line passes, so that the points lie four by four on the lines and the lines pass three by three through the points.

On each line there lie four of the points, and through each line there pass four of the planes. The four planes through a line contain all the points except the four which lie on a second line which forms a pair with the given one; and through the four points on the line there pass all the planes except the four which pass through the same second line. Further, no two of the lines intersect in a point other than one of the 40 points; and no two lines lie in a plane other than one of the 40 planes.

Moreover, with each of the 40 points may be associated a particular one of the 40 planes, so that the pairs of the nine lines which pass through the point all lie in the plane; while at the same time the nine pairs contain the whole of the 40 points, and the whole of the 40 planes pass through lines belonging to the nine pairs.

The configuration is invariant for a $G_{16\cdot 60}$, which permutes the points and planes, each transitively, while it permutes the lines in two transitive sets of 10 and 80 (5 and 40 pairs).

VI. *The G_{25920} for which the Configuration is Invariant.*

13. There are just 16 S_1-lines, which meet neither L_1 or L_2, and they form eight pairs. Any one of the 16 may be denoted by the symbol $(M_a P_b Q_c R_d)$; $(a, b, c, d = 1 \text{ or } 2)$, M_a, P_b, Q_c, R_d, being the four lines, from the original ten, which it meets; S_1 and S_2 being denoted by the symbols $(M_1 P_1 Q_1 R_1)$ and $(M_2 P_2 Q_2 R_2)$. From the pair S_1, S_2, three other pairs U_1, U_2; V_1, V_2; W_1, W_2 arise by the collineations;—

$$1, (Q_1 Q_2)(R_1 R_2), (R_1 R_2)(P_1 P_2), (P_1 P_2)(Q_1 Q_2),$$

which constitute a sub-group of order 4 of the G_{16}. These pairs are

$$\begin{pmatrix} M_1 P_1 Q_1 R_1 \\ M_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} M_1 P_1 Q_2 R_2 \\ M_2 P_2 Q_1 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_1 R_2 \\ M_2 P_1 Q_2 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_2 R_1 \\ M_2 P_1 Q_1 R_2 \end{pmatrix}.$$

Similarly, from $\bar{S}_1$ and $\bar{S}_2$—viz., $(M_1 P_1 Q_1 R_2)$, $(M_2 P_2 Q_2 R_1)$—by the collineations of the same sub-group, three other pairs $\bar{U}_1 \bar{U}_2$; $\bar{V}_1, \bar{V}_2$; $\bar{W}_1, \bar{W}_2$ arise, giving another set of 4 pairs, viz.,

$$\begin{pmatrix} M_1 P_1 Q_1 R_2 \\ M_2 P_2 Q_2 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_1 Q_2 R_1 \\ M_2 P_2 Q_1 R_2 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_1 R_1 \\ M_2 P_1 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_2 R_2 \\ M_2 P_1 Q_1 R_1 \end{pmatrix}.$$

No two lines of the first set of four pairs (or of the second set) intersect. In fact, if $M_1 P_1 Q_1 R_1$, $M_1 P_1 Q_2 R_2$ intersect, it must be in a point of M_1 or of P_1. But the a-points of both M_1 and P_1 are permuted by the collineation $(Q_1 Q_2)(R_1 R_2)$ which changes $M_1 P_1 Q_1 R_1$ into $M_1 P_1 Q_2 R_2$.

On the other hand, since the eight pairs of the two sets include all the S_1-lines which intersect neither L_1 nor L_2, while each contains four a-points, each of the 32 a-points that do not lie on L_1 or L_2 must lie on two of the 16 lines, and therefore each line of the first set must intersect four lines of the second, one from each pair.

Hence the two sets of four pairs are the only sets of four pairs of S_1-lines which contain all the 32 a-points not lying on L_1 or L_2. From the five original pairs and the 40 pairs of S_1-lines, then, just three sets of five pairs can be chosen so as to include L_1 and L_2, and to contain the whole of the 40 points. These are:—

$$\begin{array}{lllll} L_1 L_2; & M_1 M_2; & P_1 P_2; & Q_1 Q_2; & R_1 R_2; \\ L_1 L_2; & S_1 S_2; & U_1 U_2; & V_1 V_2; & W_1 W_2; \\ L_1 L_2; & \bar{S}_1 \bar{S}_2; & \bar{U}_1 \bar{U}_2; & \bar{V}_1 \bar{V}_2; & \bar{W}_1 \bar{W}_2. \end{array}$$

Similarly with each of the original five pairs, just two sets, formed each with four pairs of S_1-lines, can be constructed so as to contain the whole of the 40 points. This gives in all 11 sets, of five pairs each, from the 45 pairs of the configuration, such that each set contains all the 40 points, while one pair of each set (all five pairs of one set) belongs to the original five pairs.

14. The five pairs of S_1-lines

$$\begin{pmatrix} M_1 P_1 Q_1 R_1 \\ M_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_1 P_1 Q_1 R_1 \\ L_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_1 M_1 Q_1 R_1 \\ L_2 M_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_1 M_1 P_1 R_1 \\ L_2 M_2 P_2 R_2 \end{pmatrix}, \begin{pmatrix} L_1 M_1 P_1 Q_1 \\ L_2 M_2 P_2 Q_2 \end{pmatrix},$$

are non-intersectors. For if $(M_1 P_1 Q_1 R_1)$, $(L_1 P_1 Q_1 R)$ intersect, it must be in a point of P_1 or Q_1 or R_1, and this would mean that two of these lines lie in a plane. These five pairs therefore contain the whole of the 40 points. So also do the five pairs derived from the preceding by the collineation $[L_1 L_2]$ of the $G_{16\cdot 60}$, viz.:—

$$\begin{pmatrix} M_1 P_1 Q_1 R_1 \\ M_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_2 P_1 Q_1 R_1 \\ L_1 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_2 M_1 Q_1 R_1 \\ L_1 M_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_2 M_1 P_1 R_1 \\ L_1 M_2 P_2 R_2 \end{pmatrix}, \begin{pmatrix} L_2 M_1 P_1 Q_1 \\ L_1 M_2 P_2 Q_2 \end{pmatrix}.$$

Now a given S_1-line meets just 28 other S_1-lines and four of the original 10 lines; and its pair meets the pairs of these. Hence there are just 11 pairs of S_1-lines and one pair of the original 10 lines, which neither of a given S_1 pair intersect. For the pair S_1, S_2 or $(M_1 P_1 Q_1 R_1)$, $(M_2 P_2 Q_2 R_2)$, these 12 pairs are those which occur in the preceding sets of five, and in the set (already obtained).

$$\begin{pmatrix} M_1 P_1 Q_1 R_1 \\ M_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \begin{pmatrix} M_1 P_1 Q_2 R_2 \\ M_2 P_2 Q_1 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_1 R_2 \\ M_2 P_1 Q_2 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_2 R_1 \\ M_2 P_1 Q_1 R_2 \end{pmatrix}.$$

In each of these sets of five pairs the four pairs after the first contain all the a-points except those lying on S_1 and S_2. Hence any two pairs taken from different sets (excluding S_1 and S_2) are necessarily intersecting pairs. The three sets given are then the only sets of five pairs, including S_1 and S_2, which contain all 40 points.

Combining this with the immediately preceding result, it follows that from the 45 pairs of lines of the configuration, just three sets of five pairs each can be formed, so as to include a given pair and to contain the whole 40 points. The total number of such sets is therefore $3 \times 45/5 = 27$. Of the 27 sets, 11 contain members from a given set, including the set itself, and the remaining 16 have no members from the given set.

Each of these sets of five pairs has the property that the five pairs of lines forming it are the fixed lines of five mutually permutable collineations of order 2. This will first be proved for one of the sets of five containing an original pair, say the set

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix},$$

$$\text{or} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \begin{pmatrix} M_1 P_1 Q_1 R_1 \\ M_2 P_2 Q_2 R_2 \end{pmatrix}, \begin{pmatrix} M_1 P_1 Q_2 R_2 \\ M_2 P_2 Q_1 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_1 R_2 \\ M_2 P_1 Q_2 R_1 \end{pmatrix}, \begin{pmatrix} M_1 P_2 Q_2 R_1 \\ M_2 P_1 Q_1 R_2 \end{pmatrix}.$$

Each of the pair L_1 and L_2 is invariant for a tetrahedral sub-group of the $G_{16 \cdot 60}$, which contains the three collineations of order 2.

$$(M_1\,P_1)\,(Q_1\,R_1)\,(M_2\,P_2)\,(Q_2\,R_2),\quad (M_1\,Q_1)\,(P_1\,R_1)\,(M_2\,Q_2)\,(P_2\,R_2),$$
$$(M_1\,R_1)\,(P_1\,Q_1)\,(M_2\,R_2)\,(P_2\,Q_2),$$

say

$$A,\ B,\ AB;$$

and the collineation of order 3,

$$(P_1\,Q_1\,R_1)\,(P_2\,Q_2\,R_2),\ \text{say } C.$$

The permutations of the 10 lines forming the set considered given by these collineations are

$$A \sim (V_1 V_2)\,(W_1\,W_2),$$
$$B \sim (U_1\,U_2)\,(W_1\,W_2),$$
$$AB \sim (U_1\,U_2)\,(W_1\,W_2),$$
$$C \sim (U_1\,V_1\,W_1)\,(U_2\,V_2\,W_2).$$

From this it follows that L_1, L_2, S_1, S_2, U_1, U_2 cannot lie on a quadric; for, in virtue of the collineation C, it would also contain V_1, V_2, W_1, W_2; and the 40 points do not lie on a quadric. On the other hand, since the fixed lines of A, B and AB meet L_1, L_2, S_1, S_2, the four latter are generators of one system of a quadric, of which the fixed lines of A, B and AB, are generators of the other system. Denoting, as usual, the fixed lines of A, B, and AB, by A_1, A_2; B_1, B_2; $(AB)_1$, $(AB)_2$; U_1 and U_2 must meet both A_1 and A_2, since the collineation A leaves U_1 and U_2 unchanged. Similarly V_1 and V_2 meet both B_1 and B_2; while W_1 and W_2 meet $(AB)_1$ and $(AB)_2$. Further, since the collineation B leaves every generator of the first system unaltered, while it permutes A_1 with A_2 and U_1 with U_2, the points of intersection of A_1 with U_1, and of A_2 with U_2 must lie on a generator of the first system. Now the collineation C leaves the quadric and every generator of the first set unchanged, while it permutes cyclically the pairs A_1, A_2; B_1, B_2; $(AB)_1$, $(AB)_2$ of the other system. It also permutes cyclically the pairs U_1, U_2; V_1, V_2; W_1, W_2. Hence U_1, U_2; V_1, V_2; W_1, W_2 must join across the intersections of A_1, A_2; B_1, B_2; $(AB)_1$, $(AB)_2$ with a common pair of generators X_1, X_2 of the first system.

Hence

$$[U_1\,U_2] = [A_1\,A_2]\,[X_1\,X_2],$$
$$[V_1\,V_2] = [B_1\,B_2]\,[X_1\,X_2],$$
$$[W_1\,W_2] = [(A\,B)_1\,(A\,B)_2]\,[X_1\,X_2].$$

But $[X_1\,X_2]$ is permutable with $[A_1\,A_2]$, $[B_1\,B_2]$, $[(A\,B), (A\,B)_2]$, which are permutable among themselves. Hence $[U_1\,U_2]$, $[V_1\,V_2]$, $[W_1\,W_2]$ are mutually permutable. Now from their mode of formation the pairs S_1, S_2; U_1, U_2; V_1, V_2; W_1, W_2; are permuted among themselves by the collineations of the group

$$1, \quad (Q_1\, Q_2)(R_1\, R_2), \quad (R_1\, R_2)(P_1\, P_2), \quad (P_1\, P_2)(Q_1\, Q_2).$$

Hence $[S_1\, S_2]$, $[U_1\, U_2]$, $[V_1\, V_2]$, $[W_1\, W_2]$ are all mutually permutable; and since $[L_1\, L_2]$ permutes the lines of each pair it is permutable with each of them.

Consider now the plane $S_1\alpha$ through S_1 and an α-point on U_1. The remaining α-points on it are the points where it is met by L_1, L_2, U_2, V_1, V_2, W_1, W_2. The collineation $[S_1\, S_2]$ leaves this plane unchanged, since it leaves every point on S_2 unchanged. Moreover it permutes the pairs L_1, L_2; U_1, U_2; V_1, V_2; W_1, W_2. Hence it permutes the α-points in which $S_1\alpha$ is met by the four pairs. Similarly $[S_1\, S_2]$ permutes the α-points in the planes $S_1\beta$, $S_1\gamma$, $S_1\delta$; β, γ, δ being the other α-points on U_1. Further $[S_1\, S_2]$ leaves unchanged the eight α-points on S_1 and S_2. Hence $[S_1\, S_2]$ permutes among themselves the 40 α-points; and the same is true for the collineation of order 2, whose fixed lines are any one of the 45 pairs.

15. Consider next any one of the sets of five line-pairs which contain all the 40 points. If X_1, X_2 is any line pair of the 45, $[X_1, X_2]$ changes the five line pairs in question into another set of five which contain the 40 points. By suitably choosing X_1, X_2 it may be ensured that one of the new pairs belongs to the original five pairs. Then by the preceding paragraph the five collineations of order 2 of which the five new pairs are the fixed lines are mutually permutable; and therefore the same is true for the five pairs from which they were transformed. The five collineations of order 2, of which the five pairs are the fixed lines, generate an Abelian G_{16}. In fact, in an Abelian G_8, five collineations of order 2 cannot be chosen so that all their fixed lines are non-intersectors; and, on the other hand, there is no Abelian G_{32} of collineations of order 2. The Abelian G_{16} is contained self-conjugately in a $G_{16\cdot 60}$, for which the set of five pairs in question, and the 40 points of the configuration, are invariant. There are thus just 27 $G_{16\cdot 60}$'s for which the configuration is invariant. Each of these leaves one set of five invariant, and permutes the remaining 26 in two sets of 10 and 16. In fact, the $G_{16\cdot 60}$ for which the original set of five pairs is invariant must permute among themselves the set of 10 (§ 13), each of which contains one of the original five. Now the $G_{16\cdot 60}$ permutes the five pairs $L_1, L_2, \ldots, R_1, R_2$ transitively, and the collineation $[M_1\, M_2]$ clearly permutes the two sets (§ 13),

$$L_1\, L_2; \quad S_1\, S_2; \quad U_1\, U_2; \quad V_1\, V_2; \quad W_1\, W_2;$$

and

$$L_1\, L_2; \quad \overline{S}_1\, \overline{S}_2; \quad \overline{U}_1\, \overline{U}_2: \quad \overline{V}_1\, \overline{V}_2; \quad \overline{W}_1\, \overline{W}_2.$$

Hence these 10 sets are permuted transitively by the $G_{16\cdot 60}$. The remaining 16 sets are permuted transitively by the Abelian G_{16}, which is contained self-conjugately in the $G_{16\cdot 60}$ in question. In fact, the 16 sets into which the set of § 14, viz.:—

$$\begin{pmatrix} M_1 \, P_1 \, Q_1 \, R_1 \\ M_2 \, P_2 \, Q_2 \, R_2 \end{pmatrix}, \begin{pmatrix} L_1 \, P_1 \, Q_1 \, R_1 \\ L_2 \, P_2 \, Q_2 \, R_2 \end{pmatrix}, \begin{pmatrix} L_1 \, M_1 \, Q_1 \, R_1 \\ L_2 \, M_2 \, Q_2 \, R_2 \end{pmatrix}, \begin{pmatrix} L_1 \, M_1 \, P_1 \, R_1 \\ L_2 \, M_2 \, P_2 \, R_2 \end{pmatrix}, \begin{pmatrix} L_1 \, M_1 \, P_1 \, Q_1 \\ L_2 \, M_2 \, P_2 \, Q_2 \end{pmatrix},$$

are changed by the collineations of the original G_{16} are all distinct.

The 27 sets of five pairs then are permuted transitively by the greatest group of collineations for which the configuration is invariant, so that the order of this group must be equal to or a multiple of 27.16.60. Now any collineation which leaves each of the 27 sets unchanged must also leave each of the 45 line-pairs unchanged, and is certainly the identical collineation; *i.e.*, that which leaves every point of space unchanged.

Moreover, any collineation which leaves a single set unchanged must contain the corresponding Abelian G_{16} self-conjugately, and must therefore be a sub-group of the $G_{16\cdot 720}$ of § 6. The only sub-group of this greater than and containing the $G_{16\cdot 60}$ for which the five line-pairs are invariant, is a $G_{16\cdot 120}$, which in respect of the Abelian G_{16} is simply isomorphic with the symmetric group of five symbols. For this group, however, it may be readily verified that the points called a and a' are conjugate (the group of transformations effected on L_1 by the sub-group for which L_1 is invariant, being the octohedral and *not* the tetrahedral group), so that a is one of a set of 80 points permuted by the group. Hence the $G_{16\cdot 60}$ is the greatest group of collineations for which the 40 points and a set of five-lines-pairs is invariant.

It follows, therefore, that the order of the greatest group of collineations for which the configuration is invariant is $27.16.60 = 2^6.3^4.5$.

16. It might be anticipated that if in addition to collineations dualistic transformations were admitted, the order of the group for which the configuration is invariant would be doubled, and this is, in fact, the case.

As in § 7, an ordinary reciprocation with respect to the quadric containing L_1, L_2, M_1, M_2 leaves each of these lines unaltered, and permutes each of the pairs P_1, P_2; Q_1, Q_2; R_1, R_2. Now the $G_{16\cdot 60}$, of § 8, is contained in a $G_{16\cdot 120}$ which permutes the a-points with the a'-points, and to which belongs the collineation that permutes L_1 with M_2, L_2 with M_1, P_1 with P_2, Q_1 with Q_2, and R_1 with R_2. The reciprocation followed by this collineation is a dualistic transformation giving the permutation $(L_1 \, M_2)(L_2 \, M_1)$ of the 10 lines. This dualistic transformation is permutable with the collineation $(P_1 \, Q_1 \, R_1)(P_2 \, Q_2 \, R_2)$. The latter leaves every point of two generators K_1, K_2

of the quadric containing L_1, L_2, M_1, M_2, belonging to the system opposite to L_1, unchanged. Of these one, say K_1, meets L_1 in an a-point. Now the dualistic transformation changes the point of intersection of K_1 and L_1 into the plane containing K_1 and M_2. The point of intersection of K_1 and L_1 is one of the 40 points of the configuration; and the plane through M_2 and K_1, *i.e.*, through M_2 and an a-point on L_1, is one of the 40 planes. The configuration, therefore, is invariant for the dualistic transformation.

VII. *Identification of the* G_{25920} *with a known Simple Group.*

17. The identification of the G_{25920} of collineations thus arrived at with a known group is readily effected from a consideration of the 27 sets of five pairs, formed from the 45 pairs of lines of the collineation which are permuted by the collineations of the group. For this purpose a rather more convenient notation for the pairs is introduced. Any pair is, in fact, adequately represented by such a symbol as—

$$(a, b, c, d, e),$$

where of the five letters either one or four are zero, while the others are either 1 or 2. Thus, with this notation, the pair L_1, L_2 would be represented either by (1, 0, 0, 0, 0) or (2, 0, 0, 0, 0); while the pair $L_1, P_2 Q_1, R_1, L_2 P_1, Q_2 R_2$ is represented either by (1, 0, 2, 1, 1) or (2, 0, 1, 2, 2). Either form is derived from the other by multiplying each of the numbers by 2 and then reducing mod-3; and there is no risk of confusion between them.

With this notation the 27 sets of five are given in the following table. The first 11 are those which have a common element with the first, constructed as in §13; and the remaining 16 are those having no common element with the first; and formed by carrying out the operations of the Abelian G_{16}, for which each pair of the first set is invariant on any one of the 16, such as that given at the beginning of §14.

10000	01000	00100	00010	00001	a
10000	01111	01122	01212	01221	b
10000	01112	01121	01211	02111	c
00001	11110	11220	12120	12210	d
00001	11120	11210	12110	21110	e
01000	10111	10122	10212	10221	f
01000	10112	10121	10211	20111	g
00100	11011	11022	12012	12021	h
00100	11012	11021	12011	21011	i
00010	11101	11202	12102	12201	k
09010	11102	11201	12101	21101	l

01111	10111	11011	11101	11110	m
01111	20111	21011	21101	21110	n
01112	10112	11012	11102	11110	m'
01112	10221	12021	12201	21110	n'
01221	10221	11021	11201	11220	p
01221	10112	12012	12102	12110	q
02111	20111	11022	11202	11220	p'
02111	10111	12011	12101	12110	q'
01212	10121	12021	12101	12120	r
01212	10212	11012	11202	11210	s
01211	10122	21011	12102	12120	r'
01211	10211	11011	11201	11210	s'
01122	10211	12011	12201	12210	t
01122	10122	11022	11102	11120	u
01121	10212	12012	21101	12210	t'
01121	10121	11021	11101	11120	u'

From these 27 sets 45 triplets such as *abc* can be formed, each consisting of the three sets of five which contain one of the 45 pairs. These triplets are—

abc, adc, afg. ahi, akl, bmn, bpq, brs, btu,
$cm'n'$ $cp'q'$ $cr's'$, $ct'u'$, dmm', dpp', drr', dtt', enn',
eqq', ess', euu', fmq', fpn', fst', $fr'u$, $gm'q$; $gp'm$,
$gs't$; gru', hms', $hn'r$, $hp'u$, hqt', $im's$, inr', ipu',
$iq't$, kmu', $kn't$, $kp's$, kqr', $lm'u$, lnt', lps', $lq'r$;

where the letters denote the 27 sets of five as in the table. Every collineation of the group permutes the 27 sets, and therefore gives a permutation of the 27 letters, and these permutations must be such as also to permute the 45 triplets among themselves. But in this form the group presents itself in connection with the 27 lines on a cubic surface. The earliest investigation of the group connected with this problem is due to M. Jordan.[*] M. Jordan shows that if the 27 letters *a*, *b*, ... (the letter-notation has been chosen to agree with his) denote the 27 lines on a cubic surface, then the 45 triplets of the preceding set give the 45 triangles which can be formed from them. He further shows that the most general group of permutations of the 27 letters for which the set of triplets is invariant, is a group of order 2×25920, which contains a single sub-group (necessarily self-conjugate) of order 25920, and that this latter group is a simple group. The G_{25920} of collineations for which the configuration of points, lines, and planes is invariant is therefore a simple group isomorphic with the known simple group of the same order on which the determination of the 27 lines on a cubic surface depends.

VIII. *Some Properties of the G_{25920}.*

18. The 45 collineations of order 2 whose fixed lines are the 45 pairs of lines of the configuration, form a single conjugate set of collineations in the G_{25920}; for the group contains collineations which change any one of the 27 sets of five into any other, and in a $G_{16\cdot 60}$, for which one set of five is invariant, there are collineations changing any one pair of the five into any other.

Any one of these 45 collineations has been seen to be permutable with just 12 others, and the product of two such permutable collineations A and B is another collineation I of order 2. The fixed lines of this collineation lie on the quadric which contain the fixed lines of A and B. But it follows, from the construction of the S_1-lines, that no three non-intersecting pairs of lines chosen from the 45 can lie on a quadric. Hence I cannot be conjugate to A. Suppose now that the product of two other permutable collineations C and D of the set of 45 were I. Then the fixed lines of I would lie on two quadrics containing respectively the fixed lines of A, B and of C, D. There would therefore be two lines meeting these eight fixed lines, which is not the case. Hence I is one of a set of 270 collineations of order 2; in fact, there are 12×45 collineations of the form AB, and I occurs just twice, viz., in the forms AB and BA.

Consider next two collineations of the 45 which are not permutable, say those of which L_1, L_2 and $L_1M_1P_1Q_1$, $L_2M_2P_2Q_2$ or X_1, X_2 are the fixed lines. Denote them by L and X, and the points of intersection of L_1, X_1

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[*] 'Traité des Substitutions,' pp. 316—329.

and L_2, X_2 by a_1, a_2. Both L and X leave a_1, a_2 and the planes $L_1 X_1$, $L_2 X_2$ unchanged. Hence LX leaves every point of $a_1 a_2$ unchanged. Also L must change X_1 into another S_1-line through a_1 in the plane $L_1 X_1$, and similarly X changes L_1 into another S_1-line through a_1 in $L_1 X_1$. But there is only one other such line (§ 12). Hence

$$XLX = LXL,$$

or LX is a collineation of order 3. If Y denotes XLX, then this collineation of order 3 can be written in the three forms LX, XY, YL. Suppose now that there were other collineations of the 45, such that

$$L'X' = LX;$$

then both L′ and X′ would change $a_1 a_2$ into itself, so that $a_1 a_2$ would meet the eight fixed lines of L, X, L′, X′. This cannot be the case, and there are therefore no such collineations as L′ and X′. Hence LX is one of 480 distinct collineations. In fact there are 32×45 collineations of the forms LX, and any given one occurs just three times. This set includes with each collineation its inverse.

The line $a_1 a_2$ is one of 240, which are permuted by the group; for there are 16 lines joining any a-point on L_1 to any a-point on L_2, and of the 16×45 that so arise from the 45 pairs each occurs three times. Moreover, the sub-group for which a_1 is invariant leaves the plane $L_2 a_1'$ invariant and permutes the 12 a-points in this plane transitively. Hence the 240 lines arising from $a_1 a_2$ are permuted transitively by the group. Each of these lines is an absolutely fixed line for a collineation of order 3 and its inverse. Hence the set of 480 collineations of order 3, such as LX, is a single conjugate set.

Also the set of 270 collineations of order 2 to which I belongs is a single conjugate set. For it follows immediately, from the fact that LX is of order 3 when L and X are not permutable, that the 12 collineations of the 45 that are permutable with a given one, form a single conjugate set in the sub-group containing that given one self-conjugately.

The two conjugate sets of 45 and 270 collineations of order 2 are the only ones in the group. Any collineation of order 2 must transform some set of five pairs, say

$$L_1, L_2;\quad M_1, M_2;\quad P_1, P_2;\quad Q_1, Q_2;\quad R_1, R_2,$$

into itself. If it transforms each pair into itself, it belongs to the corresponding Abelian G_{16}, and is therefore either one of the 45 or one of the 270. If it permutes the sets it may be taken to be

$$(M_1\, P_1)\,(Q_1\, R_1)\,(M_2\, P_2)\,(Q_2\, R_2);$$

but the effect of this collineation on the five pairs

$$L_1, L_2;\quad S_1, S_2;\quad U_1, U_2;\quad V_1, V_2;\quad W_1, W_2,$$

is to transform each pair into itself. Hence, again, the collineation belongs either to the 45 or to the 270.

Another remarkable set of collineations belonging to the G_{25920} are the perspectives of order 3. It has been seen in § 12 that the 40 points all lie either on the nine lines through a or in the plane L_2a'. If any S_1-line through a meet L_2a' in α, then a, α are the fixed points of a collineation of order 3 on the line which leaves a unchanged and permutes the remaining three a-points on the line. Hence a perspective of order 3, of which a is the fixed point and L_2a' the fixed plane, permutes the 40 a-points, leaving 13 unchanged. The configuration being invariant for this perspective, it must belong to the G_{25920}. There are, then, 80 perspectives, each of order 3, having the 40 points and corresponding 40 planes for their fixed points and planes belonging to the group. No perspective can be transformed into its inverse by a collineation; so that these 80 perspectives fall into two conjugate sets of 40 each.

19. From the 27 sets of five pairs it is possible to choose 12 which contain each of 30 pairs twice and none of the remainder. Assuming the possibility of such a choice, if a, in the table of § 17, is taken for one set, then five others must come from the first 11 sets and the remainder from the other 16. Now the $G_{16\cdot 60}$ for which a is invariant permutes the last 16 sets transitively. Any one of them may therefore be taken with a. If a particular one, m, is chosen, then among the last 16 there are only five others which have a pair in common with m, viz., n, q', s', u', m'. These, then, necessarily belong to the 12. The symbols of the pairs contained in these sets have either no 2 or only one. Hence the remaining five to be chosen from the first 11 must satisfy this condition; and on a reference to the table it is seen that this can be done in just one way, viz., by taking c, g, i, l, e. The set then is uniquely determined by a and m. Of these, a can be chosen in 27 ways, and then m in 16. There are therefore $27 \times 16/12 = 36$ such sets of twelve, and they are permuted transitively by the collineations of the group. The sub-group which leaves one such set invariant is of order 720. Now both a and m are invariant for the G_{60} of § 8, generated by the collineations

$$(L_1\, M_1\, P_1\, Q_1\, R_1)\,(L_2\, M_2\, P_2\, Q_2\, R_2),$$
$$(L_1\, M_1)\,(P_1\, Q_1)\,(L_2\, M_2)\,(P_2\, Q_2).$$

Hence the pair of sets a, m is one of either 6 or 12 pairs, permuted transitively by the G_{720} for which the set of twelve is invariant. If it were one of 12 there would be two pairs containing a, and their other members would be either invariant for or permuted by the G_{60}. Now the G_{60} is simple, and hence each other member would be invariant for the G_{60}. But a and m are the only two members of the set of twelve which are invariant for the G_{60}. Hence the pair a, m is invariant for a G_{120} and is one of six such pairs for the G_{720}, and the remaining five pairs are c, n; g, q'; i, s'; l, u'; e, m', these being permuted by the G_{60}. A collineation which leaves each of these pairs invariant is found to leave each of the 30 pairs of lines which enter in the set of twelve invariant, and is therefore the identical collineation. Hence the G_{720} is simply isomorphic with a permutation group on 6 symbols. It is therefore the symmetric group in six symbols, and the G_{120} for which the pair a, m is invariant is the symmetric group of five symbols.

Moreover, the groups arrived at thus are the only sub-groups of the G_{25920} simply isomorphic with the symmetric group of the 6 symbols.* Any such sub-group must permute the 27 sets of five intransitively, in groups of degree 6, 10, 12 or 15. The only possible combinations are 12, 15, the case considered above; or 6, 6, 15. Now if six sets of five undergo the permutations of the symmetric group of 6 symbols, each pair of sets must contain a common line-pair, or else no pair of sets contains a common line-pair. A reference to the table shows that neither case is possible.

By an extension of the above reasoning it may be shown that for any sub-group of the G_{25920} which permutes the 27 sets of five intransitively, the smallest number of sets transitively permuted among themselves must be 1, 2, 3, 9 or 12. A set of nine contains each of six pairs of lines three times, and 27 other pairs each once. Such a set is

$$\begin{matrix} a, & b, & c, \\ d, & m, & m', \\ e, & n, & n'. \end{matrix}$$

20. For the sub-groups of the G_{25920} of indices 40 and 45 there are simple geometrical invariants; that for the sub-groups of index 40 consisting of one of the 40 points and its associated plane; while one of the 45 pairs of lines of the configuration is invariant for a sub-group of index 45. For the sub-groups of indices 27 and 36 there are no such simple invariants.

For the $G_{16\cdot 60}$ of index 27, the corresponding 5 line pairs form an invariant figure; these are the non-intersecting fixed lines of five mutually permutable collineations of order 2. If P_0 is any point of space, and P_1, P_2, P_3, P_4, P_5, the points with which P_0 is changed by the five collineations of order 2, then $P_0, P_1, P_2, P_3, P_4, P_5$ lie in a plane; this is, in fact, a well-known property of what is called the 16_6 configuration. The five mutually permutable collineations of order 2 then co-ordinate with every point of space a definite plane passing through it; *i.e.*, they define a linear complex, and this complex is invariant for the $G_{16\cdot 60}$. To the 27 sub-groups conjugate with the $G_{16\cdot 60}$ there correspond 27 linear complexes which are permuted by the collineations of the group. These complexes are such that from them may be formed 45 sets of three, such that for each set of three the complex planes corresponding to the same point meet in a line; or in other words, such that the three of a set are not linearly independent. This property is characteristic, and enables the 27 complexes to be constructed from any six which are linearly independent. To effect this determination I represent the complexes which correspond to the 12 sets of five of the preceding paragraph as follows:—

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* *Cf.* Dickson, 'Lond. Math. Soc. Proc.,' vol. 1, New Series, p. 283.

$$\begin{array}{cccccc} a, & c, & g, & i, & l, & e, \\ A_0, & A_1, & A_2, & A_3, & A_4, & A_5, \end{array} \qquad \begin{array}{cccccc} m, & n, & q', & s', & u', & m', \\ B_0, & B_1, & B_2, & B_3, & B_4, & B_5; \end{array}$$

and I denote the fact that C can be expressed linearly in terms of A and B by the notation

$$C \sim AB;$$

while the condition that a greater number of linear complexes are not independent is represented by

$$A, B, C, D, \ldots \sim 0.$$

Then if the remaining complexes are denoted by the letters used for the corresponding sets of five, the 45 linear relations between the complexes are given by the following tables:—

$$b \sim A_0\,A_1 \sim B_0\,B_1,$$
$$f \sim A_0\,A_2 \sim B_0\,B_2,$$
$$h \sim A_0\,A_3 \sim B_0\,B_3,$$
$$k \sim A_0\,A_4 \sim B_0\,B_4,$$
$$d \sim A_0\,A_5 \sim B_0\,B_5,$$

$$p' \sim A_1\,B_2 \sim A_2\,B_1,$$
$$r' \sim A_1\,B_3 \sim A_3\,B_1,$$
$$t' \sim A_1\,B_4 \sim A_4\,B_1,$$
$$n' \sim A_1\,B_5 \sim A_5\,B_1,$$
$$t \sim A_2\,B_3 \sim A_3\,B_2,$$

$$r \sim A_2\,B_4 \sim A_4\,B_2,$$
$$q \sim A_2\,B_5 \sim A_5\,B_2,$$
$$p \sim A_3\,B_4 \sim A_4\,B_3,$$
$$s \sim A_3\,B_5 \sim A_5\,B_3,$$
$$u \sim A_4\,B_5 \sim A_5\,B_4,$$

are the first 30. With suitable weights to the complexes, these relations involve

$$A_0+B_0 = A_1+B_1 = A_2+B_2 = A_3+B_3 = A_4+B_4 = A_5+B_5 = C, \text{ say.}$$

Hence $$B_i = C-A_i;$$

and in terms of the A's b, f, h, k, d, are given by

$$A_0-A_1, \quad A_0-A_2, \quad A_0-A_3, \quad A_0-A_4, \quad A_0-A_5.$$

The remaining 15 relations consist of five sets of three of the form

$$-A_0+A_1,\ A_2,\ A_2,\ B_4,\ B_5 \sim 0,$$
$$-A_0+A_2,\ A_1,\ A_3,\ B_4,\ B_5 \sim 0,$$
$$-A_0+A_3,\ A_1,\ A_2,\ B_4,\ B_5 \sim 0.$$

These can only be satisfied by

$$2C = -A_0+A_1+A_2+A_3+A_4;$$

and then the previous relations determine the remaining 16 complexes uniquely.

The complete set of 27 which verify the required relations are then given in terms of six independent ones (as A_0, A_1, A_2, A_3, A_4, A_5 certainly are) as follows:—

$$A_0,\quad A_1,\quad A_2,\quad A_3,\quad A_4,\quad A_5;\quad (6)$$

$$\begin{array}{ll} -3A_0+A_1+A_2+A_3+A_4+A_5, & -A_0-A_1+A_2+A_3+A_4+A_5, \\ -A_0+A_1-A_2+A_3+A_4+A_5, & -A_0+A_1+A_2-A_3+A_4+A_5, \\ -A_0+A_1+A_2+A_3-A_4+A_5, & -A_0+A_1+A_2+A_3+A_4-A_5;\quad (6) \end{array}$$

$$A_0-A_1,\quad A_0-A_2,\quad A_0-A_3,\quad A_0-A_4,\quad A_0-A_5;\quad (5)$$

and
$$-A_0-A_1-A_2+A_3+A_4+A_5,$$

with nine similar sets, each having three plus and three minus signs (10).

The sub-group of index 36 and order 720, considered in § 19, permutes transitively the six following pairs of the 27 linear complexes, viz.,

$$\begin{array}{ll} A_0, & -3A_0+A_1+A_2+A_3+A_4+A_5; \\ A_1, & -A_0-A_1+A_2+A_3+A_4+A_5; \\ A_2, & -A_0+A_1-A_2+A_3+A_4+A_5; \\ A_3, & -A_0+A_1+A_2-A_3+A_4+A_5; \\ A_4, & -A_0+A_1+A_2+A_3-A_4+A_5; \\ A_5, & -A_0+A_1+A_2+A_3+A_4-A_5. \end{array}$$

Now there is clearly just one linear complex, viz.,

$$-A_0+A_1+A_2+A_3+A_4+A_5,$$

which can be expressed linearly in terms of each of the six pairs that are permuted by the sub-group. Hence the sub-group must leave this linear complex invariant. The simplest geometrical invariant for the sub-groups of index 36 is therefore a linear complex.*

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* *Cf.* H. Burkhardt, "Hyperelliptische Modulfunctionen III," 'Math. Ann.,' vol. 41, pp. 321—326.